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Distributed control of multi-agent systems with random parameters and a major agent[☆]

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A B S T R A C T

Distributed control of the multi-agent systems involving a major agent and a large number of minor agents is investigated in this paper. There exist Markov jump parameters in the dynamic equation and random parameters in the index functions. The major agent has salient impact on others. Each minor agent merely has tiny influence, while the average effect of all the minor agents is not negligible, which plays a significant role in the evolution and performance index of each agent. Besides the state of the major agent, each minor agent can only access to the information of its state and parameters. Based on the mean field (MF) theory, a set of distributed control laws is designed. By the probability limit theory, the uniform stability of the closed-loop system and the upper bound of the corresponding index values are obtained. Via a numerical example, the consistency of the MF estimation and the influence of the initial state values and parameters on the index values are demonstrated.

1. Introduction

The topic of multi-agent systems (MASs) is attracting more and more interest and attention due to its wide application backgrounds in economics, social science, biology and engineering (Akkermans, Schreinemakers, & Kok, 2004; Guessoum, Rejeb, & Durand, 2004; Reynolds, 1987; Schurr et al., 2005; Sun & Naveh, 2004; Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995). In MASs, each agent merely relies on the local information to make decisions. Thus, a natural problem arising is how to design the distributed control laws based on each agent's information pattern and performance index. In the case of large populations, due to the high computation complexity, it is generally hard to be implemented in practical applications. To conquer this drawback, Huang, Caines, and Malhame (2003, 2007); Huang, Malhamé, and Caines (2006) developed the Nash Certainty Equivalence Methodology based on the mean field (MF) theory, with which they gave distributed ε-Nash equilibrium strategies for the game problem of large population MASs coupled via discounted costs. Li and Zhang (2008a,b) considered the case where agents are coupled via their stochastic long run time-average indices, and obtained an asymptotical Nash equilibrium. Wang and Zhang (in press) considered the mean field games of MASs where agents were coupled by nonlinear indices and the structure parameters were independent Markov chains. The identical idea can also be found in Lasry and Lions (2006, 2007), Weintraub, Benkard, and Van Roy (2005, 2008), Yin, Mehta, Meyn, and Shanbhag (2010). Weintraub et al. (2005, 2008) presented an MF approximation method when studying discrete-time stochastic games, and proposed the notion of oblivious equilibrium which can approximate Markov perfect equilibrium. Lasry and Lions (2006, 2007) introduced an MF game model, and provided a limit partial differential equation. Yin et al. (2010) investigated the synchronization of coupled nonlinear oscillators in a game-theoretic framework, and gave a deterministic partial differential equation model for mean field approximation in the stochastic systems.

The above-mentioned papers mainly consider the model involving a large number of agents with the equal influence. However, in practice the agents may have different influences. In this kind of systems, an important model with wide backgrounds is the system involving one or several major agents and many minor agents, such as a market consisting of several large companies and a large number of small companies. (Fudenberg, Levine, & Pesendorfer, 1998; Levine & Pesendorfer, 1995) studied when the influence of each minor agent is negligible. Huang (2010) investigated continuous-time stochastic dynamic games of large population systems with a major player, and gave a set of ε-Nash equilibrium strategies for the systems under some consistency conditions. Benkard, Jeziorksi, and Weintraub (2010) considered oblivious equilibria with dominant firms in industry models.

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In practice, another factor to be considered is the uncertainty of environment. For instance, the change rates of prices in the financial market in different time slots may be very different. A powerful tool depicting the abrupt environmental changes is Markov jump models. To be specific, owing to drastic variation of environment, the system may take on different "modes" at different moments, and jump among these "modes" according to a Markov chain. For the research results in this direction, readers are referred to (Costa, Fragoso, & Marques, 2005; Ji & Chizeck, 1990; Mariton, 1990; Sworder, 1969; Wonham, 1970).

In this paper, we investigate the model with a major agent and a large number of minor agents. The major agent plays an important role in the evolution of the system. Each minor agent merely has tiny influence, while the average effect of all the minor agents is not negligible. Compared with the previous works, the model in the paper has the following features. (1) There are Markov jump parameters in the dynamic equation and random parameters in the index function for each agent. (2) Each minor agent only knows its initial state. In the previous work, each agent is required to know the statistical expectation of the initial values of all the minor agents. Thus, the model in the paper has broader application backgrounds. For instance, consider a market consisting of a large company and many small companies in an environment with abrupt changes. Different companies may have different initial assets and anticipated earnings. The large company, which possesses abundant capital and market information, has great impact on the market. However, the small companies with tiny influence, may have difficulty in obtaining the statistical information for all the other small companies.

For this kind of models, we design distributed control laws based on the MF theory. Since the parameters in all agents' indices may be different and each minor agent does not know the statistical expectation of the initial state of all the minor agents, each agent may have different estimates for the state average of the minor agents. On the other hand, since each agent is affected significantly by the major agent, the state average of all the minor agents depends on the major agent. Thus, even when the number of agents grows to infinity, the aggregation effect of all the minor agents is not a deterministic quantity, but a stochastic process depending on the major agent's state. The dependence on the major agent's state and diversity of the MF estimation functions bring us more difficulty to the analysis of the closed-loop system. We first prove that the MF estimation function approximates the state average of all the minor agents by exploiting the structure of the system and the property of Markov chains. And then, regarding the MF estimation function of the major agent and its state as a high dimensional variable, from the combined evolutionary equations we obtain their uniform boundedness in the average sense and the uniform stability of the closed-loop system. Finally, we give an upper bound for the corresponding index values and show that the index values of all the closed-loop agents under the distributed control converge to the optimal index values under the centralized control, as the variance of the parameters of the minor agents' indices decreases to 0 and the number of agents grows to infinity.

The remainder of this paper is organized as follows. In Section 2, we describe the model and basic assumptions. In Section 3, we first give the optimal centralized control and the corresponding index values, and then design a set of distributed control laws based on the MF theory. In Section 4, we analyze the stability of the closed-loop system and the optimality of the distributed control. Firstly, we prove that the MF estimation function approximates to the state average of all the minor agents. Then we obtain the uniform stability of the closed-loop systems and an upper bound for the corresponding index values under the distributed control, and compare it with the optimal index values under the centralized control. In Section 5, we analyze how the values of some parameters in the index functions affect the stability and optimality of the closed-loop system. In Section 6, through a numerical example, we verify the consistency of the MF estimation, and demonstrate the influence of parameter values and initial state values on the index values. In Section 7, we give some concluding remarks for the paper.

The following notations will be used in the paper. For a given vector or matrix X, X^T denotes the transpose of X; $\mathrm{tr}(X)$ denotes the trace of the square matrix X; $\|X\|$ denotes the Euclidean vector norm or matrix norm induced by the Euclidean vector norm of X. I_n denotes an n-dimensional identity matrix; $\otimes$ denotes the Kronecker product. For a sequence of matrices $A_j, j = 1, \ldots, m$, $\mathrm{diag}(A_j)$ denotes the block diagonal matrix with A_j in the diagonal and zero elsewhere. For a given random variable (r.v.) ξ on the probability space $(\Omega, \mathcal{F}, P)$, $E\xi$ denotes the mathematical expectation of ξ. For a given set collection $\mathcal{C}$, $\sigma(\mathcal{C})$ denotes the σ algebra generated by $\mathcal{C}$. For a family of $\mathbb{R}^n$-values r.v.s $\{\xi_\lambda, \lambda \in \Lambda\}$, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ algebra $\sigma\{[\xi_\lambda \in B], B \in \mathcal{B}^n, \lambda \in \Lambda\}$, where $\mathcal{B}^n$ is an n dimensional Borel σ algebra.

2. Problem description

Consider the MAS described by the following dynamics:

$$
\begin{aligned}
x_0(k+1) &= f_0(\theta_k, x_0(k)) + u_0(k) + F_0(\theta_k)x^{(N)}(k) \\
&\quad + D_0(\theta_k)w_0(k+1), \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
x_i(k+1) &= f_i(\theta_k, x_i(k)) + u_i(k) + F(\theta_k)x^{(N)}(k) \\
&\quad + G(\theta_k)x_0(k) + D(\theta_k)w_i(k+1), \quad 1 \le i \le N, \qquad (2)
\end{aligned}
$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^r$ and $w_i(k) \in \mathbb{R}^d$, $0 \le i \le N$ are the state, the input and the stochastic disturbance of the agent i, respectively. Agent 0 denotes the major agent, and the others are the minor agents. $x^{(N)}(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(k)$ is the state average of all the minor agents. $\{\theta_k\}$ is a discrete-time ergodic Markov chain taking value in $S = \{1, 2, \ldots, m\}$ with the transition probability matrix $P = \{p_{ij}, i, j = 1, \ldots, m\}$ and the stationary distribution $\pi = \{\pi_j, j = 1, \ldots, m\}$. $f_i : S \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Borel measurable function, $F_0(\cdot), D_0(\cdot), F(\cdot), G(\cdot)$ and $D(\cdot)$ are real matrices with proper dimensions.

The indices of N+1 agents are described by

$$
J_0^N(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\left\|x_0(k+1) - H_0x^{(N)}(k) - \alpha_0\right\|^2, \qquad (3)
$$

$$
\begin{aligned}
J_i^N(u) &= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\left\|x_i(k+1) - (\bar{H}_0x_0(k)\right. \\
&\quad \left. + \; Hx^{(N)}(k) + \alpha_i)\right\|^2, \qquad (4)
\end{aligned}
$$

where $\{\alpha_i, 0 \le i \le N\}$ is a sequence of independent n dimensional random variables (r.v.s). $H_0, \bar{H}_0$ and H are matrices with proper dimensions. $u = (u_0, u_1, \ldots, u_N)$.

In the model (1)–(4), the major agent 0 possesses significant impact on the evolution behavior and performance indices of all the other agents. The influence of each minor agent is tiny, but the average effect of all the minor agents is not negligible. This kind of model is widely existent in many fields, e.g., the market consisting of one or several large companies and a large number of small companies in economics, and the system involving the government and mass in social science.

Remark 2.1. The model (1)–(4) may be used to roughly describe the market consisting of a large company and many small companies in an environment with abrupt changes. x_0 denotes the output level of the large company; $x_i, 1 \leq i \leq N$, denotes the output level of the ith small company; $u_i, 0 \leq i \leq N$, denotes the input of the ith company; $x^{(N)}$ denotes the average of outputs of all the small companies. During the operation of the market, each company may anticipate that its averaged output gained in a relatively long time period attains some value, which depends on x_0 and $x^{(N)}$. Indeed, the tracking-type indices (3)–(4) are based on the works of Huang et al. (2007) and Lambson (1984), which investigated the dynamical production output planning. Under the assumption that the demand increases with the number of companies, the market price is roughly regarded as a linear function of x_0 and $x^{(N)}$. We assume that each company tries to keep its output level approximately in proportion to the price of market since an increasing price calls for more supplies of the product and vice versa. Thus, each company will adjust its output to "track" some value depending on x_0 and $x^{(N)}$ by minimizing a quadratic penalty term.

For the convenience of citations, here we list some assumptions to be used in the paper.

(A1) $\{\{w_i(k), k \geq 0\}, 0 \leq i \leq N\}$ is a family of independent d dimensional white noise sequences defined on the probability space $(\Omega, \mathcal{F}, P)$,

$$E[w_i(k)] = 0, \qquad E[w_i(k)w_i^T(j)] = \delta_{kj} I_d, \quad 0 \leq i \leq N,$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j; \\ 0, & \text{otherwise.} \end{cases}$$

(A2) The initial state values $\{x_{i0}, 1 \leq i \leq N\}$ are independent r.v.s with an identical mathematical expectation. $E\|x_{00}\|^2 < \infty$; $\max_{1\leq i\leq N} E\|x_{i0} - Ex_{i0}\|^2 = \delta_0$. $\{x_{i0}, 0 \leq i \leq N\}$, $\{w_i(k), 0 \leq i \leq N\}$ and $\{\theta_k\}$ are independent of each other.

(A3) $E\alpha_i = \alpha, \ 1 \leq i \leq N, \max_{1\leq i\leq N} E\|\alpha_i - \alpha\|^2 = \delta, E\|\alpha_0\|^2 < \infty$.

Throughout this paper, on the information available to each agent, we will assume that the state and parameters of each agent are known to itself. Ex_{10} and α are not available to the minor agents, but available to the major agent 0. The state of the major agent is available to all the minor agents.

Remark 2.2. The above assumptions have wide practical backgrounds. For instance, in a market consisting of a large company and many small companies, the companies may have different initial assets and anticipated outputs. Since the number of companies is quite large, the initial assets of small companies can be regarded as random samples of the same distribution function. The large company, which possesses abundant capital, can capture the statistical information of the initial assets and anticipated outputs of all the small companies. Meanwhile, with high transparency and considerable publicity, the large company is the focus of concern, and its operational status can be captured by the small companies.

Remark 2.3. In (1)–(4), if $\theta_k \equiv \theta, \alpha_i \equiv \alpha$, then the model degenerates to the one of Huang (2010). Our work is not a trivial extension of Huang (2010). On the one hand, Huang (2010) considered the case of continuous time; we investigate the case of discrete time, and give a set of easy-be-verified distributed control laws (see Remark 3.2). On the other hand, the information assumptions are different. In Huang (2010), Ex_{10} is assumed to be known to each agent, but here, such information is not required.

3. Design of control laws

First, we provide two groups of control sets:

$$\mathcal{U}_{g,i} = \left\{ u \middle| u(k) \in \sigma\left\{ \bigcup_{0\leq i\leq N} \sigma(\alpha_i, x_i(j), \theta_j, 0 \leq j \leq k) \right\} \right\},$$

$$\mathcal{U}_{l,i} = \left\{ u \middle| u(k) \in \sigma(\alpha_i, x_i(j), x_0(j), \theta_j, 0 \leq j \leq k) \right\}, \quad 0 \leq i \leq N.$$

$\mathcal{U}_{g,i}$ is called the global-measurement-based control set, and $\mathcal{U}_{l,i}$ is called the local-measurement-based control set. For each agent i, when the admissible control set is $\mathcal{U}_{g,i}$, the corresponding control is called centralized control; when the admissible control set is $\mathcal{U}_{l,i}$, the corresponding control is called distributed control. The main objective of the paper is to design distributed control for the large population MAS (1)–(4).

To inspire the design of the distributed control, we first give the centralized optimal control and the corresponding index values.

Theorem 3.1. *For the system (1)–(4), if Assumptions (A1) and (A2) hold, then it follows that*

$$\inf_{\{u_l \in \mathcal{U}_{g,l}, 0\leq l\leq N\}} J_0^N(u) = \sum_{j=1}^{m} \pi_j \mathrm{tr}(D_0(j)D_0^T(j)), \tag{5}$$

$$\inf_{\{u_l \in \mathcal{U}_{g,l}, 0\leq l\leq N\}} J_i^N(u) = \sum_{j=1}^{m} \pi_j \mathrm{tr}(D_j D_j^T), \quad 1 \leq i \leq N, \tag{6}$$

where $D_j = D(j)$. In particular, if we take the control laws as follows

$$\bar{u}_0(k) = (H_0 - F_0(\theta_k))x^{(N)}(k) + \alpha_0 - f_0(\theta_k, x_0(k)), \tag{7}$$

$$\begin{aligned} \bar{u}_i(k) \ = \ & (\bar{H}_0 - G(\theta_k))x_0(k) + (H - F(\theta_k))x^{(N)}(k) \\ & + \alpha_i - f_i(\theta_k, x_i(k)), \quad 1 \leq i \leq N, \end{aligned} \tag{8}$$

then $\bar{u}_i \in \mathcal{U}_{g,i}, 0 \leq i \leq N$, and the corresponding index values are

$$J_0^N(\bar{u}) = \sum_{j=1}^{m} \pi_j \mathrm{tr}(D_0(j)D_0^T(j)), \tag{9}$$

$$J_i^N(\bar{u}) = \sum_{j=1}^{m} \pi_j \mathrm{tr}(D_j D_j^T), \quad 1 \leq i \leq N. \tag{10}$$

Proof. See Appendix. □

Remark 3.1. Noticing in the model (1)–(4), each agent has an individual performance index, this model may be regarded as a game problem. Different from the case of the single agent, the meaning of the optimality of game problems is very diversified. However, Nash equilibrium and Pareto-optimality are the two most frequently encountered. A set of strategies (control) is called a Nash equilibrium if no player (agent) has incentive to change its strategy unilaterally; a set of strategies is Pareto-optimal, if there is no other strategy that makes one better off without making someone else worse off (Başar & Olsder, 1999; Fudenberg & Tirole, 1991). From (5)–(6), Theorem 3.1 and its proof, one can see that the set of strategies (7)–(8) is not only a Nash equilibrium but also Pareto-optimal, although by the Folk Theorem (Dutta, 1995), (1)–(4) may have many Nash equilibria.

Indeed, (7)–(8) is also a subgame perfect equilibrium, which is a refinement of the Nash equilibrium.

Definition 3.2 (Fudenberg & Tirole, 1991)**.** A set of strategies $u = (u_0, \dots, u_N)$ is called a subgame perfect equilibrium, if for every history $h^k = \{(u_0(j), \dots, u_N(j)), j \le k-1\}$, the restriction $u|h^k$ to $G(h^k)$ is a Nash equilibrium of $G(h^k)$, where $u|h^k$ is the restriction of u to the histories consistent with h^k and $G(h^k)$ is the game from stage k on with history h^k.

Theorem 3.3. *For the system (1)–(4), if* (A1) *and* (A2) *hold, then* $(\bar u_0, \dots, \bar u_N)$ *given by (7)–(8) is a subgame perfect equilibrium of (1)–(4).*

Proof. Noticing $(\bar u_0, \dots, \bar u_N)$ is a set of feedback strategies, and (3) and (4) are time-averaged indices over the infinite horizon, by the proof of Theorem 3.1 (especially, (A.1) and (A.2) in Appendix), we get that for every $h^k = \{(u_0(j), \dots, u_N(j)), j \le k-1\}$, $u|h^k$ is a Nash equilibrium of $G(h^k)$. □

Due to the particular structure and information limitation, generally speaking, it is hard to get such a distributed control that can reach the same optimal performance as that the centralized optimal control can. Thus, it is natural to consider a suboptimal distributed control. In this paper, we first estimate the state average of all the minor agents, and then construct a set of distributed control laws implementable in practical applications. Finally, we analyze the stability and optimality of the closed-loop system.

Now, we proceed to design the distributed control laws based on the MF theory.[2]

First, we construct the auxiliary system described by

$$\begin{aligned}\check x_0(k+1) &= f_0(\theta_k, \check x_0(k)) + \check u_0(k) + F_0(\theta_k) g(k)\\ &\quad + D_0(\theta_k) w_0(k+1),\end{aligned} \tag{11}$$

$$\begin{aligned}\check x_i(k+1) &= f_i(\theta_k, \check x_i(k)) + \check u_i(k) + F(\theta_k) g(k)\\ &\quad + G(\theta_k)\check x_0(k) + D(\theta_k) w_i(k+1), \quad 1 \le i \le N,\end{aligned} \tag{12}$$

with index functions

$$\check J_0(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\left\|\check x_0(k+1) - H_0 g(k) - \alpha_0\right\|^2, \tag{13}$$

$$\begin{aligned}\check J_i(u) &= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|\check x_i(k+1) - (\bar H_0 \check x_0(k)\\ &\quad + Hg(k) + \alpha_i)\|^2, \quad 1 \le i \le N.\end{aligned} \tag{14}$$

In (11)–(14), $g(k)$, $k \ge 0$, is a random sequence, which, called as MF estimation function, is regarded as the estimate for the state average of all the minor agents. In this case, the optimal tracking control is

$$\check u_0(k) = [H_0 - F_0(\theta_k)] g(k) + \alpha_0 - f_0(\theta_k, \check x_0(k)), \tag{15}$$

$$\begin{aligned}\check u_i(k) &= [\bar H_0 - G(\theta_k)]\check x_0(k) + [H - F(\theta_k)] g(k) + \alpha_i\\ &\quad - f_i(\theta_k, \check x_i(k)), \quad 1 \le i \le N.\end{aligned} \tag{16}$$

Applying (15) and (16) into the dynamic equations (11) and (12), the closed-loop equations can be written as

$$\check x_0(k+1) = H_0 g(k) + \alpha_0 + D_0(\theta_k) w_0(k+1), \tag{17}$$

$$\begin{aligned}\check x_i(k+1) &= \bar H_0 \check x_0(k) + Hg(k) + \alpha_i\\ &\quad + D(\theta_k) w_i(k+1), \quad 1 \le i \le N\end{aligned} \tag{18}$$

which implies

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \check x_i(k+1) &= \bar H_0 \check x_0(k) + Hg(k) + \frac{1}{N}\sum_{i=1}^{N}\alpha_i\\ &\quad + \frac{1}{N}\sum_{i=1}^{N} D(\theta_k) w_i(k+1).\end{aligned} \tag{19}$$

By the law of large numbers (Chow, 1997; Chung, 2001) we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\alpha_i = \alpha, \quad \text{a.s.,}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} D(\theta_k) w_i(k+1) = E[D(\theta_k) w_i(k+1)] = 0, \quad \text{a.s.}$$

Thus, from the MF theory, the Nash Certainty Equivalence Methodology (Huang et al., 2006) and (19), the MF estimation function $g(k)$ should satisfy the following recursive equation:

$$g(k+1) = \bar H_0 x_0(k) + Hg(k) + \alpha.$$

Since the information and parameters of initial values available to each agent are different, they may have different estimates for the state average of the minor agents. Agent 0's MF estimation function is

$$g^*(k+1) = Hg^*(k) + \bar H_0 x_0(k) + \alpha, \qquad g^*(0) = Ex_{10}. \tag{20}$$

Agent i's MF estimation function is

$$\begin{aligned}&g_i^*(k+1) = Hg_i^*(k) + \bar H_0 x_0(k) + \alpha_i,\\ &g_i^*(0) = x_{i0}, \quad 1 \le i \le N.\end{aligned} \tag{21}$$

In summary, we obtain the following distributed control laws:

$$u_0^*(k) = (H_0 - F_0(\theta_k)) g^*(k) + \alpha_0 - f_0(\theta_k, x_0(k)), \tag{22}$$

$$\begin{aligned}u_i^*(k) &= (\bar H_0 - G(\theta_k)) x_0^*(k) + (H - F(\theta_k)) g^*(k)\\ &\quad + \alpha_i - f_i(\theta_k, x_i(k)), \quad 1 \le i \le N.\end{aligned} \tag{23}$$

Remark 3.2 (Huang, 2010)**.** Investigated continuous-time distributed games for large population systems involving a major player. They gave a distributed strategy based on the Nash Certainty Equivalence Methodology. But, to ensure the existence of distributed strategies, some not-easy-be-verified consistency conditions are needed. Here we get rid of these conditions and give distributed control laws through the explicit recursive equations (20)–(23).

4. Analysis of the closed-loop system

Applying the control laws (22) and (23) to (1) and (2), one can get the following closed-loop systems:

$$\begin{aligned}x_0(k+1) &= H_0 g^*(k) + \alpha_0 + F_0(\theta_k)(x^{(N)}(k) - g^*(k))\\ &\quad + D_0(\theta_k) w_0(k+1),\end{aligned} \tag{24}$$

$$\begin{aligned}x_i(k+1) &= g_i^*(k+1) + F(\theta_k)(x^{(N)}(k) - g_i^*(k))\\ &\quad + D(\theta_k) w_i(k+1), \quad 1 \le i \le N,\end{aligned} \tag{25}$$

where $g^*(k)$ and $g_i^*(k)$, $i = 1, \dots, N$, are given by (20) and (21).

To ensure the stability and optimality of the closed-loop system, we need the following assumptions:

(A4) H, M and Γ are stable, i.e., all of their eigenvalues are inside the unit circle, where

$$M = \begin{pmatrix} H & \bar H_0 \\ H_0 & 0 \end{pmatrix}, \qquad \Gamma = (P^T \otimes I_{n^2})\,\mathrm{diag}\{F(j)\otimes F(j)\}.$$

[2] The mean field (MF) approach (theory) is a relatively standard technique, which is primarily used in physics and chemistry (e.g., the derivation of Boltzmann equations in the kinetic gas theory). In addition to the applications in distributed controls mentioned in the introduction, it is widely used in the study on the limit behavior of Markov decision processes (Gast & Gaujal, 2011; Gast, Gaujal, & Le Boudec, 2010; Kurtz, 1978), evolutionary games (Benaim & Weibull, 2003; Tembine, Le Boudec, El-Azouzi, & Altman, 2009) etc.

Remark 4.1. It is a relatively standard assumption that H and Γ are stable (e.g. Costa et al., 2005). When $\bar{H}_0 = 0$ or $H_0 = 0$, the stability of M is equivalent to the stability of H. Roughly speaking, that M is stable ensures that the major agent's MF estimation function is bounded in the average sense; that H and M are stable ensures that the minor agents' MF estimation functions are uniformly bounded in the average sense; that Γ is stable ensures that the major agent's MF estimation function approximates to the state average of all the minor agents.

To analyze the closed-loop system, we need the following lemma.

Lemma 1. *Let $A_j \in \mathbb{R}^{n\times n}, j = 1,\dots,m$. $\{W(k), k \geq 0\}$ is a d-dimensional stochastic sequence, where*

$$EW(k) = 0, \qquad EW(j)W(k) = r\delta_{jk}I_d, \quad r \geq 0.$$

$\{\theta_k\}$ is an ergodic Markov chain taking value in $S = \{1, 2, \dots, m\}$ with the transition probability matrix P and the stationary distribution π. If $(P^T \otimes I_{n^2})\mathrm{diag}\{A_i \otimes A_i\}$ is stable, then there exists the constant C_0, such that for the following stochastic difference equation

$$X(k+1) = A(\theta_k)X(k) + D(\theta_k)W(k+1),$$

where $E\|X(0)\|^2 < \infty$, the solution satisfies

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|X(k)\|^2 \leq rC_0 \sum_{j=1}^{m} \pi_j \mathrm{tr}(D_j^T D_j).$$

Proof. See Appendix. □

First, we give an approximation result.

Theorem 4.1. *For the system (1)–(2) with the indices (3)–(4), if Assumptions* (A1)–(A4) *hold, then under the control laws (22) and (23), there is a constant C_1 such that*

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|x^{(N)}(k) - g^*(k)\|^2 \leq \frac{C_1}{N}.$$

Proof. By Assumption (A1) we have

$$E\left[\frac{1}{N}\sum_{i=1}^{N} w_i(k)\right] = \frac{1}{N}\sum_{i=1}^{N} Ew_i(k) = 0,$$

$$\begin{aligned} E\left[\frac{1}{N}\sum_{i=1}^{N} w_i(k)\frac{1}{N}\sum_{j=1}^{N} w_j^T(k)\right] &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i(k)w_j^T(k) \\ &= \frac{N}{N^2}I_d = \frac{1}{N}I_d. \end{aligned}$$

When $k \neq l$, it follows that

$$E\left[\frac{1}{N}\sum_{i=1}^{N} w_i(k)\frac{1}{N}\sum_{j=1}^{N} w_j^T(l)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} Ew_i(k)w_j^T(l) = 0.$$

Let $\eta_N(k) = x^{(N)}(k) - \frac{1}{N}\sum_{i=1}^{N} g_i(k)$. Then, (25) can be rewritten as

$$\eta_N(k+1) = F(\theta_k)\eta_N(k) + D(\theta_k)\left[\frac{1}{N}\sum_{i=1}^{N} w_i(k+1)\right],$$

which together with (A4) and Lemma 1 gives

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|\eta_N(k)\|^2 \leq \frac{C_0}{N}\sum_{j=1}^{m} \mathrm{tr}(\pi_j D_j D_j^T). \tag{26}$$

From (21) we have

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N} g_i^*(k+1) &= \bar{H}_0 x_0(k) + H\left[\frac{1}{N}\sum_{i=1}^{N} g_i^*(k)\right] \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\alpha_i, \end{aligned}$$

$$\frac{1}{N}\sum_{i=1}^{N} g_i^*(0) = \frac{1}{N}\sum_{i=1}^{N} x_{i0},$$

and hence, from (20),

$$\begin{aligned} &\frac{1}{N}\sum_{i=1}^{N} g_i^*(k+1) - g^*(k+1) \\ &= H\left[\frac{1}{N}\sum_{i=1}^{N} g_i^*(k) - g^*(k)\right] + \frac{1}{N}\sum_{i=1}^{N}\alpha_i - \alpha \\ &= H^{k+1}\left(\frac{1}{N}\sum_{i=1}^{N} x_{i0} - Ex_{10}\right) + \sum_{j=0}^{k} H^j\left(\frac{1}{N}\sum_{i=1}^{N}\alpha_i - \alpha\right). \end{aligned}$$

Since H is stable, by Horn and Johnson (1990) there exists $\lambda_h \in (0, 1)$ and $C_h > 0$ such that for any integer $k \geq 1$, $\|H^k\| \leq C_h\lambda_h^k$. Noticing

$$E\left\|\frac{1}{N}\sum_{i=1}^{N} x_{i0} - Ex_{10}\right\|^2 = \frac{\sum_{i=1}^{N} E\|x_{i0} - Ex_{i0}\|^2}{N^2} \leq \frac{\delta_0}{N},$$

$$E\left\|\frac{1}{N}\sum_{i=1}^{N}\alpha_i - \alpha\right\|^2 \leq \frac{N\max_{1\leq i\leq N} E\|\alpha_i - \alpha\|^2}{N^2} = \frac{\delta}{N},$$

where $\delta_0 = \max_{1\leq i\leq N} E\|x_{i0} - Ex_{i0}\|^2$ and $\delta = \max_{1\leq i\leq N} E\|\alpha_i - \alpha\|^2$, we have

$$\begin{aligned} &E\left\|\frac{1}{N}\sum_{i=1}^{N} g_i^*(k+1) - g^*(k+1)\right\|^2 \\ &\leq \frac{2}{N}\|H^{k+1}\|^2 E\|x_{i0} - Ex_{i0}\|^2 + 2\left(\sum_{j=0}^{k}\|H^j\|\right)^2\frac{\delta}{N} \\ &\leq \frac{2}{N}C_h^2\lambda_h^{2k+2}\delta_0 + \frac{2\delta}{N}\left(\sum_{j=0}^{k} C_h^2\lambda_h^j\right)^2 \\ &\leq \frac{2}{N}\left(C_h^2\delta_0 + \frac{C_h^2\delta}{(1-\lambda_h)^2}\right). \end{aligned}$$

This together with (26) implies

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\|x^{(N)}(k) - g^*(k)\|^2 \\ &\leq \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\left[2\|\eta_N(k)\|^2 + 2\left\|\frac{1}{N}\sum_{i=1}^{N} g_i(k) - g^*(k)\right\|^2\right] \\ &\leq \frac{2C_0}{N}\sum_{j=1}^{m}\mathrm{tr}(\pi_j D_j D_j^T) + \frac{4}{N}\left(C_h^2\delta_0 + \frac{C_h^2\delta}{(1-\lambda_h)^2}\right) \triangleq \frac{C_1}{N}. \quad \square \end{aligned}$$

Remark 4.2. The above theorem shows that the major agent's MF estimation function approximates the state average of all the minor agents. From the proof of the theorem, it follows that the law of large numbers plays an important role in the approach of mean field approximation.

We now show the uniform stability of the closed-loop system.

Theorem 4.2. *For the system (1)–(2) with the indices (3)–(4), if Assumptions (A1)–(A4) hold, then under the control laws (22) and (23), the closed-loop system has the following property:*

$$\sup_{N\geq 1}\max_{0\leq i\leq N}\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x_i(k)\|^2<\infty.$$

To prove this, we need two lemmas.

Lemma 2. *Let $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^n$. Then the following inequality holds:*

$$\|x+y\|^2\leq a\|x\|^2+b\|y\|^2,$$

where $a>1$, $b>1$, $1/a+1/b=1$.

Proof. By the Cauchy inequality,

$$2x^Ty\leq(a-1)\|x\|^2+1/(a-1)\|y\|^2.$$

Hence,

$$\begin{aligned}\|x+y\|^2&=\|x\|^2+2x^Ty+\|y\|^2\leq a\|x\|^2\\&\quad+(1+1/(a-1))\|y\|^2=a\|x\|^2+b\|y\|^2.\quad\square\end{aligned}$$

Lemma 3. *For the system (1)–(2) with the indices (3)–(4), if Assumptions (A1)–(A4) hold, then under the control laws (22) and (23), there exist the constants C_2 and C_3 independent of N such that*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E(\|g^*(k)\|^2+\|x_0(k)\|^2)<C_2,\tag{27}$$

$$\max_{1\leq i\leq N}\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|g_i^*(k)\|^2<C_3.\tag{28}$$

Proof. Let

$$y(k)=\begin{pmatrix}g^*(k)\\x_0(k)\end{pmatrix},$$

$$q(k)=\begin{pmatrix}\alpha\\F_0(\theta_k)(x^{(N)}(k)-g^*(k))+D_0(\theta_k)w_0(k+1)+\alpha_0\end{pmatrix}.$$

Then, (20) and (24) can be rewritten as

$$y(k+1)=My(k)+q(k).\tag{29}$$

Since M is stable, from Horn and Johnson (1990), Guo (1993), there is a matrix norm induced by the vector norm $\|\cdot\|_0$, which is also denoted as $\|\cdot\|_0$ for simplicity, such that $\|M\|_0<1$. Furthermore, from (29) and Lemma 2 it follows that

$$\begin{aligned}E\,\|y(k+1)\|_0^2&\leq a\|M\|_0^2E\|y(k)\|_0^2+bE\|q(k)\|_0\\&\leq a\|M\|_0^2E\|y(k)\|_0^2+b\Big[\|\alpha\|_0^2+D_0(\theta_k)w_0(k+1)\\&\quad+E\|F_0(\theta_k)(x^{(N)}(k)-g^*(k))+\alpha_0\|_0^2\Big]\\&\leq a\|M\|_0^2E\|y(k)\|_0^2+b\|\alpha\|_0^2\\&\quad+3b\Big[\max_{1\leq j\leq m}\|F_0(j)\|_0^2E\|x^{(N)}(k)-g^*(k)\|_0^2\\&\quad+\max_{1\leq j\leq m}\|D_0(j)\|_0^2+E\|\alpha_0\|_0^2\Big],\end{aligned}$$

where $a>1$ and $a\|M\|_0^2<1$. Hence,

$$\begin{aligned}&\frac{1}{T}\sum_{k=0}^{T}E\|y(k+1)\|_0^2\\&\leq a\|M\|_0^2\frac{1}{T}\sum_{k=0}^{T}E\|y(k)\|_0^2+b\|\alpha\|_0^2\\&\quad+3b\Big[\max_{1\leq j\leq m}\|F_0(j)\|_0^2\frac{1}{T}\sum_{k=0}^{T}E\|x^{(N)}(k)-g^*(k)\|_0^2\\&\quad+E\|\alpha_0\|_0^2+\max_{1\leq j\leq m}\|D_0(j)\|_0^2\Big].\end{aligned}\tag{30}$$

Noticing

$$\begin{aligned}\frac{1}{T}\sum_{k=0}^{T}E\|y(k+1)\|_0^2&=\frac{1}{T}\sum_{k=0}^{T}E\|y(k)\|_0^2\\&\quad+\frac{1}{T}E\|y(T+1)\|_0^2-\frac{1}{T}E\|y(0)\|_0^2,\end{aligned}$$

by (30) we have

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|y(k)\|_0^2\\&\leq\frac{b}{1-a\|M\|_0^2}\Big[\|\alpha\|_0^2+3E\|\alpha_0\|_0^2+3\max_{1\leq j\leq m}\|D_0(j)\|_0^2\\&\quad+3\max_{1\leq j\leq m}\|F_0(j)\|_0^2\limsup_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\|x^{(N)}(k)-g^*(k)\|_0^2\Big].\end{aligned}$$

Notice that the norms in the finite dimensional linear space are equivalent, and

$$E\,\|y(k)\|^2=E\|g^*(k)\|^2+E\|x_0(k)\|^2.$$

Then, by Theorem 4.1, (27) holds.

We now prove (28). By (20) and (21) we have

$$\begin{aligned}g_i^*(k+1)-g^*(k+1)&=H[g_i^*(k)-g^*(k)]+\alpha_i-\alpha\\&=H^{k+1}(x_{i0}-Ex_{i0})+\sum_{j=0}^{k}H^j(\alpha_i-\alpha).\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|g_i^*(k+1)-g^*(k+1)\|^2\\&\leq\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\Big[2\|H^{k+1}\|^2E\|x_{i0}-Ex_{i0}\|^2\\&\quad+2\Big(\sum_{j=0}^{k}\|H^j\|\Big)^2E\|\alpha_i-\alpha\|^2\Big]\\&\leq\limsup_{T\to\infty}\frac{2C_h^2}{T(1-\lambda_h^2)}\max_{1\leq i\leq N}E\|x_{i0}-Ex_{i0}\|^2\\&\quad+\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\frac{2C_h^2\max_{1\leq i\leq N}E\|\alpha_i-\alpha\|^2}{(1-\lambda_h)^2}\\&=\frac{2C_h^2\delta}{(1-\lambda_h)^2}.\end{aligned}\tag{31}$$

This together with (27) gives

$$\max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|g_i^*(k)\|^2$$
$$\le \limsup_{T\to\infty} \frac{2}{T}\sum_{k=0}^{T} E\|g^*(k)\|^2$$
$$+ \limsup_{T\to\infty} \frac{2}{T}\sum_{k=0}^{T} E\|g_i^*(k)-g^*(k)\|^2$$
$$\le 2C_2 + \frac{4C_h^2\delta}{(1-\lambda_h)^2} \triangleq C_3.$$

That is, (28) holds. □

Proof of Theorem 4.2. From (25), (28), (31) and Theorem 4.1 it follows that

$$\max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|x_i(k+1)\|^2$$
$$= \max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T}\Big[E\|F(\theta_k)(x^{(N)}(k)-g_i^*(k))$$
$$+ g_i^*(k+1)\|^2 + E\|D(\theta_k)w_i(k+1)\|^2\Big]$$
$$\le 3\max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\Big\{\max_{1\le j\le m}\|g_i^*(k+1)\|^2$$
$$+ \max_{1\le j\le m}\|F(j)\|^2\|g^*(k)-g_i^*(k)\|^2$$
$$+ \|F(j)\|^2\|x^{(N)}(k)-g^*(k)\|^2\Big\} + \sum_{j=1}^{m}\pi_j \mathrm{tr}(D_jD_j^T)$$
$$\le 3\left\{\max_{1\le j\le m}\|F(j)\|^2\left[\frac{C_1}{N}+\frac{2C_h^2\delta}{(1-\lambda_h)^2}\right]+C_3\right\}$$
$$+ \sum_{j=1}^{m}\pi_j \mathrm{tr}(D(j)D^T(j)). \quad (32)$$

This together with (27) implies the theorem. □

We now show the optimality.

Theorem 4.3. *For the system (1)–(2) with the indices (3)–(4), if Assumptions* (A1)–(A4) *hold, then under the distributed control laws (22) and (23) (i.e., $u_i^* \in \mathcal{U}_{l,i}$), the corresponding index values satisfy*

$$J_0^N(u^*) \le \sum_{j=1}^{m}\pi_j \mathrm{tr}(D_0(j)D_0^T(j)) + O\left(\frac{1}{N}\right), \quad (33)$$

$$J_i^N(u^*) \le \sum_{j=1}^{m}\pi_j \mathrm{tr}(D(j)D^T(j)) + O\left(\frac{1}{\sqrt{N}}\right) + \frac{\sum_{j=1}^{m}\pi_j\|F(j)-H\|^2C_h^2\delta}{(1-\lambda_h)^2}. \quad (34)$$

Remark 4.3. Comparing (34) with (10), we can see that the above index value of each minor agent is at most larger than the optimal index value under the centralized control (Pareto-optimal index value) with the size

$$\sum_{j=1}^{m}\pi_j\|F(j)-H\|^2C_h^2\delta/(1-\lambda_h)^2 + O(1/\sqrt{N}).$$

Thus, as the largest variance δ of the r.v. sequences $\{\alpha_i\}$ decreases to 0 and the number of agents grows to ∞, the index value tends to the optimal one reached by centralized control.

Proof of Theorem 4.3. By Assumption (A1), (3), (24), (32) and Theorem 4.1, we have

$$J_0^N(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|(F_0(\theta_k)-H_0)[x^{(N)}(k)-g^*(k)]$$
$$+ D_0(\theta_k)w_0(k+1)\|^2$$
$$= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|(F_0(\theta_k)-H_0)[x^{(N)}(k)-g^*(k)]\|^2$$
$$+ E\|D_0(\theta_k)w_0(k+1)\|^2$$
$$\le \max_{1\le l\le m}\|F_0(l)-H_0\|^2\frac{C_1}{N} + \sum_{j=1}^{m}\pi_j \mathrm{tr}(D_0(j)D_0^T(j))$$
$$= \sum_{j=1}^{m}\pi_j \mathrm{tr}(D_0(j)D_0^T(j)) + O\left(\frac{1}{N}\right).$$

Thus, (33) holds.

From Assumption (A1), (4), (21), (25), (32) and Theorem 4.1 it follows that

$$J_i^N(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|(F(\theta_k)-H)[x^{(N)}(k)-g_i^*(k)]$$
$$+ D(\theta_k)w_i(k+1)\|^2$$
$$= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\Big\{\|(F(\theta_k)-H)[x^{(N)}(k)-g_i^*(k)]\|^2$$
$$+ \|D(\theta_k)w_i(k+1)\|^2\Big\}$$
$$= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\Big\{\|F(\theta_k)-H\|^2\Big[\|x^{(N)}(k)-g^*(k)\|^2$$
$$+ 2\|x^{(N)}(k)-g^*(k)\|\,\|g^*(k)-g_i^*(k)\|$$
$$+ \|g^*(k)-g_i^*(k)\|^2\Big]\Big\} + \sum_{j=1}^{m}\pi_j \mathrm{tr}(D(j)D^T(j))$$
$$\triangleq I_1 + I_2 + I_3 + \sum_{j=1}^{m}\pi_j \mathrm{tr}(D(j)D^T(j)), \quad (35)$$

where

$$I_1 = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\big[\|F(\theta_k)-H\|^2\|x^{(N)}(k)-g^*(k)\|^2\big],$$

$$I_2 = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\big[\|F(\theta_k)-H\|^2\|g^*(k)-g_i^*(k)\|^2\big],$$

$$I_3 = 2\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\big[\|F(\theta_k)-H\|^2$$
$$\times \|x^{(N)}(k)-g^*(k)\|\,\|g^*(k)-g_i^*(k)\|\big].$$

We now calculate I_1, I_2 and I_3, respectively. By Theorem 4.1 we have

$$I_1 \le \max_{1\le j\le m} E\|F(j)-H\|^2 \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\|x^{(N)}(k)-g^*(k)\|^2$$
$$\le \max_{1\le j\le m}\|F(j)-H\|^2\frac{C_1}{N}.$$

Noticing there exists $\lambda_h \in (0,1)$ and $C_h > 0$ such that for any integer $k \geq 1$, $\|H^k\| \leq C_h\lambda_h^k$, by Assumptions (A2)–(A3), (31) and the ergodicity of $\{\theta_k\}$ we have

$$\begin{aligned}
I_2 &= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\left[\|F(\theta_k)-H\|^2\left\|H^k(x_{i0}-Ex_{i0})\right.\right.\\
&\quad\left.\left.-\sum_{j=0}^{k-1}H^j(\alpha_i-\alpha)\right\|^2\right]\\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\|F(\theta_k)-H\|^2\\
&\quad\times\left\{\|H^{2k}\|E\|x_{i0}-Ex_{i0}\|^2+\left(\sum_{j=0}^{k-1}\|H^j\|\right)^2E\|\alpha_i-\alpha\|^2\right.\\
&\quad\left.+2E\left[\|H^k\|\,\|x_{i0}-Ex_{i0}\|\sum_{j=0}^{k-1}\|H^j\|\,\|\alpha_i-\alpha\|\right]\right\}\\
&\leq \max_{1\leq j\leq m}\|F(j)-H\|^2\limsup_{T\to\infty}\frac{C_h^2E\|x_{i0}-Ex_{i0}\|^2}{T(1-\lambda_h^2)}\\
&\quad+\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|F(\theta_k)-H\|^2\left[\frac{C_h(1-\lambda_h^k)}{1-\lambda_h}\right]^2\\
&\quad\times E\|\alpha_i-\alpha\|^2+2\max_{1\leq j\leq m}\|F(j)-H\|^2\frac{C_h^2\lambda_h^k}{1-\lambda_h}\\
&\quad\times\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\left(E\|x_{i0}-Ex_{i0}\|^2E\|\alpha_i-\alpha\|^2\right)^{\frac{1}{2}}\\
&\leq \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|F(\theta_k)-H\|^2\frac{C_h^2\delta}{(1-\lambda_h)^2}\\
&\quad+2\max_{1\leq j\leq m}\|F(j)-H\|^2\limsup_{T\to\infty}\frac{C_h^2}{T(1-\lambda_h)^2}(\delta_0\delta)^{\frac{1}{2}}\\
&\leq \frac{C_h^2\delta\sum_{j=1}^{m}\pi_j\|F(j)-H\|^2}{(1-\lambda_h)^2}. \qquad (36)
\end{aligned}$$

By the Schwarz inequality, Theorem 4.1 and (31) we have

$$\begin{aligned}
I_3 &\leq 2\max_{1\leq j\leq m}\|F(j)-H\|^2\\
&\quad\times\left(\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x^{(N)}(k)-g^*(k)\|^2\right)^{1/2}\\
&\quad\times\left(\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|g^*(k)-g_i^*(k)\|^2\right)^{1/2}\\
&\leq 2\max_{1\leq j\leq m}\|F(j)-H\|^2\left(\frac{2C_1C_h^2\delta}{N(1-\lambda_h)^2}\right)^{1/2}\\
&= O(1/\sqrt{N}).
\end{aligned}$$

This together with (35)–(36) leads to (34). Thus, the theorem is true. □

In Remark 4.3, the index values under the distributed control $\{u_0^*,\dots,u_N^*\}$ have been compared with the Pareto-optimal index values. We now consider the equilibrium property of $\{u_0^*,\dots,u_N^*\}$. First, we give the definition of a weak Nash equilibrium (Başar & Olsder, 1999).

Definition 4.4. A set of control $\{u_i \in \mathcal{U}_{l,i}, 0\leq i\leq N\}$ is called an ε-Nash equilibrium with respect to the set of index functions $\{J_i^N, 0\leq i\leq N\}$, if there exists $\varepsilon \geq 0$ such that for any $0\leq i\leq N$,

$$J_i^N(u_i,u_{-i}) \leq \inf_{u_i'\in\mathcal{U}_{g,i}} J_i^N(u_i',u_{-i})+\varepsilon,$$

where $u_{-i}=\{u_0,\dots,u_{i-1},u_{i+1},\dots,u_N\}$, $J_i^N(u_i,u_{-i})=J_i^N(u)$.

The following theorem shows that $\{u_0^*,\dots,u_N^*\}$ is a weak Nash equilibrium.

Theorem 4.5. *For the system (1)–(2) with the indices (3)–(4), if Assumptions (A1)–(A4) hold, then $\{u_0^*,\dots,u_N^*\}$ is an ε-Nash equilibrium, where*

$$\varepsilon=\sum_{j=1}^{m}\pi_j\|F(j)-H\|^2C_h^2\delta/(1-\lambda_h)^2+O(1/\sqrt{N}).$$

Proof. Noticing

$$\inf_{\{u_l\in\mathcal{U}_{g,l},0\leq l\leq N\}} J_i(u) \leq \inf_{u_i'\in\mathcal{U}_{g,i}} J_i^N(u_i',u_{-i}),$$

by Theorems 3.1 and 4.3, it follows that

$$\begin{aligned}
J_i^N(u_i^*,u_{-i}^*) &= J_i^N(u^*) \leq \inf_{u_i'\in\mathcal{U}_{g,i}} J_i^N(u_i',u_{-i}^*)\\
&\quad+\frac{\sum_{j=1}^{m}\pi_j\|F(j)-H\|^2C_h^2\delta}{(1-\lambda_h)^2}+O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

which gives the conclusion of the theorem. □

Next, we will consider the case where the "reference signals" in the index functions (3) and (4) are more general, i.e. the indices of $N+1$ agents are described by

$$J_0^N(u)=\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\left\|x_0(k+1)-\Phi(x^{(N)}(k))\right\|^2, \qquad (37)$$

$$J_i^N(u)=\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\left\|x_i(k+1)-\Psi(x_0(k),x^{(N)}(k))\right\|^2, \qquad (38)$$

where Φ and Ψ are general functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume $x_{i0}\equiv Ex_{10}$, $1\leq i\leq N$. By constructing the auxiliary system, we get that the MF estimate function

$$g^*(k+1)=\Psi(x_0(k),g^*(k)),\qquad g^*(0)=x_{10},$$

and the following distributed control laws:

$$u_0^*(k)=\Phi(g^*(k))-F_0(\theta_k)g^*(k)-f_0(\theta_k,x_0(k)), \qquad (39)$$

$$\begin{aligned}
u_i^*(k) &= \Psi(x_0(k),g^*(k))-G(\theta_k)x_0(k)-F(\theta_k)g^*(k)\\
&\quad-f_i(\theta_k,x_i(k)),\quad 1\leq i\leq N. \qquad (40)
\end{aligned}$$

From this we have the following closed-loop system:

$$\begin{aligned}
x_0(k+1) &= \Phi(g^*(k))+F_0(\theta_k)(x^{(N)}(k)-g^*(k))\\
&\quad+D_0(\theta_k)w_0(k+1), \qquad (41)
\end{aligned}$$

$$\begin{aligned}
x_i(k+1) &= \Psi(x_0(k),g^*(k))+F(\theta_k)(x^{(N)}(k)-g_i^*(k))\\
&\quad+D(\theta_k)w_i(k+1),\quad 1\leq i\leq N. \qquad (42)
\end{aligned}$$

Assume

(A4′) there exist constants L_1, L_2 and L_3, such that for any $x, y, z\in\mathbb{R}^n$,

$\|\Phi(x)-\Phi(y)\|\leq L_1\|x-y\|$,

$\|\Psi(x,y)-\Psi(x,z)\|\leq L_2\|y-z\|$,

$\|\Psi(x,y)\|\leq L_3$.

Then, we have the following results of stability and optimality, whose proofs are similar to those of Theorems 4.2 and 4.3, and thus, omitted here.

Theorem 4.6. *For the system (1)–(2) with the indices (37)–(38), if Assumptions* (A1)–(A3) *and* (A4$'$) *hold, then under the distributed control laws (39) and (40), the corresponding closed-loop system and index values satisfy*

$$\sup_{N\geq 1}\max_{0\leq i\leq N}\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x_i(k)\|^2<\infty,$$

$$J_0^N(u^*)\leq\sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0(j)D_0^T(j))+O\left(\frac{1}{N}\right),$$

$$J_i^N(u^*)\leq\sum_{j=1}^{m}\pi_j\mathrm{tr}(D(j)D^T(j))+O\left(\frac{1}{N}\right),$$

for $1\leq i\leq N$, and the set of control laws (39) and (40) is an ε-Nash equilibrium, where $\varepsilon=O(1/N)$.

Remark 4.4. It is worth pointing out that Assumption (A4$'$) is merely a sufficient condition to ensure the stability and optimality of the closed-loop system. More relaxed conditions, including the sufficient and necessary conditions, need to be explored further.

Remark 4.5. Indeed, it is mainly attributed to the law of large numbers that mean field approaches work. The approach of this paper can be used to deal with the case where the index of the agent $i(1\leq i\leq N)$ is a general function of x_i, x_0 and $x^{(N)}$. For instance, for the system (1)–(2), the indices of agents are, respectively,

$$J_0(u)=\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|\phi(x_0(k+1),x^{(N)}(k))\|^2,$$

$$J_i(u)=\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|\psi(x_i(k+1),x_0(k),x^{(N)}(k))\|^2,$$

where ϕ and ψ are general functions. For this model, the distributed control laws can be designed by replacing $x^{(N)}$ with the iterative random function g^*. However, compared with the tracking case, it is more complicated to design control and analyze the closed-loop system. Particularly, when constructing the distributed control laws, a minimization problem of nonlinear functions will arise and need to be tackled.

5. Influence of the index parameters on the closed-loop system

In this section, we consider the case of $\bar{H}_0=0$. In this case, from (21) and (25) one can see that the closed-loop minor agents are not affected by the major agent anymore. Thus, the subsystem consisting of all the minor agents is equivalent to such an MAS whose agents are with equal influence.

We now analyze the case where the parameters H and α take different values.

Case I. H is stable. In this case, $M=\begin{pmatrix}H & 0\\ H_0 & 0\end{pmatrix}$ is also stable. Thus, Theorem 4.3 still holds. In particular, when $\alpha_i\equiv\alpha$ and $\theta_j\equiv 1$, the index value (34) is degenerated to the result of Li and Zhang (2008b, Theorem 4.3).

Case II. H is unstable and $\alpha_i\neq 0$. Since $\bar{H}=0$, (21) becomes

$$g_i^*(k+1)=Hg_i^*(k)+\alpha_i,\qquad g_i^*(0)=x_{i0},\quad 1\leq i\leq N,$$

or equivalently,

$$g_i^*(k)=H^kx_{i0}+\sum_{j=0}^{k-1}H^j\alpha_i.$$

Noticing that H is unstable, we known that there is α_i such that $\limsup_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\|g_i^*(k)\|^2=\infty$. Thus, the stability of the closed-loop system cannot be ensured.

Case III. $H=I_n$ and $\alpha_i\equiv 0$. In this case, (20) and (21) can be rewritten as

$$g^*(k)\equiv Ex_{10},\qquad g_i^*(k)\equiv x_{i0},\quad 1\leq i\leq N.\tag{43}$$

By (22) and (23) we get the following distributed control laws:

$$u_0^*(k)=(H_0-F_0(\theta_k))Ex_{10}+\alpha_0-f_0(\theta_k,x_0(k)),\tag{44}$$

$$\begin{aligned}u_i^*(k)&=(I_n-F(\theta_k))x_{i0}-f_i(\theta_k,x_i(k))-G(\theta_k)x_0(k),\\&\quad 1\leq i\leq N,\end{aligned}\tag{45}$$

and hence, the closed-loop system

$$\begin{aligned}x_0(k+1)&=H_0Ex_{10}+\alpha_0+F_0(\theta_k)(x^{(N)}(k)-Ex_{10})\\&\quad+D_0(\theta_k)w_0(k+1),\end{aligned}\tag{46}$$

$$\begin{aligned}x_i(k+1)&=x_{i0}+F(\theta_k)(x^{(N)}(k)-x_{i0})\\&\quad+D(\theta_k)w_i(k+1),\quad 1\leq i\leq N.\end{aligned}\tag{47}$$

This together with Theorem 4.1 leads to

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x^{(N)}(k)-Ex_{10}\|^2\leq\frac{C_1}{N}.\tag{48}$$

We now study the uniform stability of the closed-loop system.

Theorem 5.1. *For the system (1)–(2) with the indices (3)–(4), if $\bar{H}_0=0, H=I_n, \alpha_i\equiv 0$ and Assumptions* (A1)–(A4) *hold, then under the control laws (44) and (45), the closed-loop system has the following property:*

$$\sup_{N\geq 1}\max_{0\leq i\leq N}\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x_i(k)\|^2<\infty.\tag{49}$$

Proof. By Assumption (A1), (46) and (48) we have

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x_0(k+1)\|^2\\&\leq 2\max_{1\leq j\leq m}\|F_0(j)\|^2\frac{C_1}{N}+2\|H_0Ex_{10}+\alpha_0\|^2\\&\quad+\sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0(j)D_0^T(j));\end{aligned}$$

and by Assumption (A1), (47) and (48),

$$\begin{aligned}&\max_{1\leq i\leq N}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}E\|x_i(k+1)\|^2\\&=\max_{1\leq i\leq N}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\Big[E\Big\|F(\theta_k)(x^{(N)}(k)-Ex_{i0}\\&\quad+Ex_{i0}-x_{i0})+x_{i0}\Big\|^2+E\|D(\theta_k)w_i(k+1)\|^2\Big]\\&\leq 3\max_{1\leq j\leq m}\|F(j)\|^2\left(\frac{C_1}{N}+\delta_0\right)+3\max_{1\leq i\leq N}E\|x_{i0}\|^2\\&\quad+\sum_{j=1}^{m}\pi_j\mathrm{tr}(D_jD_j^T).\end{aligned}$$

Thus, (49) holds. $\square$

We now study the optimality.

Theorem 5.2. *For the system (1)–(2) with the indices (3)–(4), if $\bar{H}_0 = 0, H = I_n, \alpha_i \equiv 0$ and Assumptions (A1)–(A4) hold, then under the control laws (44) and (45), the corresponding indices satisfy:*

$$J_0^N(u^*) \le \sum_{j=1}^m \pi_j \mathrm{tr}(D_0(j)D_0^T(j)) + O\left(\frac{1}{N}\right), \tag{50}$$

$$J_i^N(u^*) \le \sum_{j=1}^m \pi_j \mathrm{tr}(D(j)D^T(j)) \tag{51}$$

$$+ \sum_{j=1}^m \pi_j \|F(j) - I_n\|^2 \delta_0 + O\left(\frac{1}{\sqrt{N}}\right), \tag{52}$$

where $\delta_0 = \max_{1\le i\le N} E\|x_{i0} - Ex_{i0}\|^2$.

Proof. Similar to (33), from (46) and (48) we can get (50). We now prove (51). By Assumption (A1), (4), (47) and (48) we have

$$\begin{aligned}
J_i^N(u^*) &= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^T E\big\|(F(\theta_k) - I_n)[x^{(N)}(k) - x_{i0}] \\
&\quad + D(\theta_k)w_i(k+1)\big\|^2 \\
&= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^T E\big\|(F(\theta_k) - I_n)[x^{(N)}(k) - Ex_{10} \\
&\quad + Ex_{10} - x_{i0}]\big\|^2 + \|D(\theta_k)w_i(k+1)\|^2 \\
&= \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^T E\Big\{\|F(\theta_k) - I_n\|^2\big[\|x^{(N)}(k) - Ex_{10}\|^2 \\
&\quad + \|x_{i0} - Ex_{10}\|^2 + 2\|x^{(N)}(k) - Ex_{10}\|\,\|x_{i0} - Ex_{10}\|\big]\Big\} \\
&\quad + \sum_{j=1}^m \pi_j \mathrm{tr}(D(j)D^T(j)) \\
&\le \max_{1\le j\le m} \|F(j) - I_n\|^2 \frac{C_1}{N} + \sum_{j=1}^m \pi_j \|F(j) - I_n\|^2\delta_0 \\
&\quad + \sum_{j=1}^m \pi_j \mathrm{tr}(D(j)D^T(j)) + 2\max_{1\le j\le m}\|F(j) - I_n\|^2 \\
&\quad \times \left(\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^T E\|x^{(N)}(k) - Ex_{10}\|^2\|x_{i0} - Ex_{10}\|^2\right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^m \pi_j\|F(j) - I_n\|^2\delta_0 + \sum_{j=1}^m \pi_j \mathrm{tr}(D(j)D^T(j)) \\
&\quad + O\big(1/\sqrt{N}\big).
\end{aligned}$$

That is, (51) holds. □

Remark 5.1. Li and Zhang (2008b) considered the model involving N agents with the equal role. Each agent was required to know the statistical expectation of the initial values of all the agents. Here we get rid of this requirement and only assume that each agent knows the initial value of itself. This should be very welcome in real applications. The cost that we should pay for is that only sub-optimality can be obtained, although the index value (51) coincides with the one given by Li and Zhang (2008b) in the case of $\delta_0 = 0$, where the initial value of each agent equals its expectation.

6. Numerical examples

We now use a numerical example to illustrate the main result of this paper, including the consistency of MF estimation, the sub-optimality of distributed control laws and the influence of the parameters α_i and the initial values x_{i0} on the index values of the minor agents.

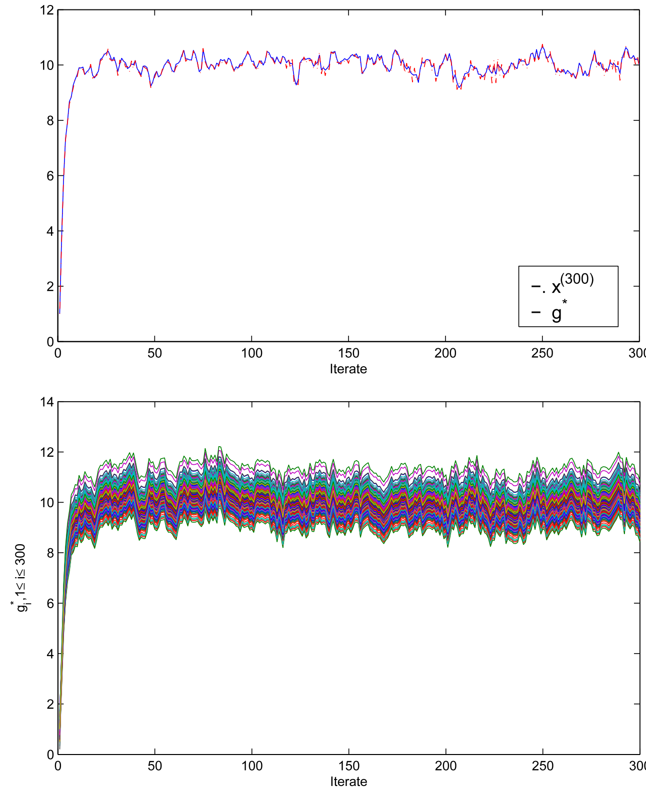

Fig. 1. Trajectories of $x^{(300)}$, g^* and g_i^*, $i = 1, \ldots, 300$.

The dynamic equations of $N + 1$ agents are given by

$$\begin{aligned}
x_0(k+1) &= 3x_0(k) + u_0(k) + F_0(\theta_k)x^{(N)}(k) \\
&\quad + D_0(\theta_k)w_0(k+1), && (53)\\
x_i(k+1) &= 2x_i(k) + u_i(k) + F(\theta_k)x^{(N)}(k) + 0.25x_0(k) \\
&\quad + D(\theta_k)w_i(k+1), \quad 1\le i\le N, && (54)
\end{aligned}$$

where $F_0(1) = -0.5, F_0(2) = -0.4, F(1) = -0.96, F(2) = -0.9, D_0(1) = 1.2, D_0(2) = 0.8, D(1) = 1$ and $D(2) = 0.6$. $\{\theta_k\}$ is a Markov chain taking value in $\{1, 2\}$ with the transition probability matrix

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix},$$

and the stationary distribution (0.5 0.5). $\{w_i(k),\ 0 \le i \le N\}$ is a Gaussian white noise sequence with the normal distribution $N(0, 1)$. Let the initial value $x_{00} = 5, \{x_{i0},\ i = 1, \ldots, N\}$ be independent and identically distributed (i.i.d.) r.v.s with the normal distribution $N(1, 0.2)$. The index functions, respectively, are given by

$$J_0^N(u) = \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^T E\left\|x_0(k+1) - x^{(N)}(k) - 5\right\|^2, \tag{55}$$

$$\begin{aligned}
J_i^N(u) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^T E \\
&\quad \times \left\|x_i(k+1) - (\bar{H}_0x_0(k) + Hx^{(N)}(k) + \alpha_i)\right\|^2, && (56)
\end{aligned}$$

where $\bar{H}_0 = 0.2, H = 0.5$, and $\{\alpha_i, i = 1, \ldots, N\}$ is i.i.d. with $\alpha_i \sim N(2, 0.09)$.

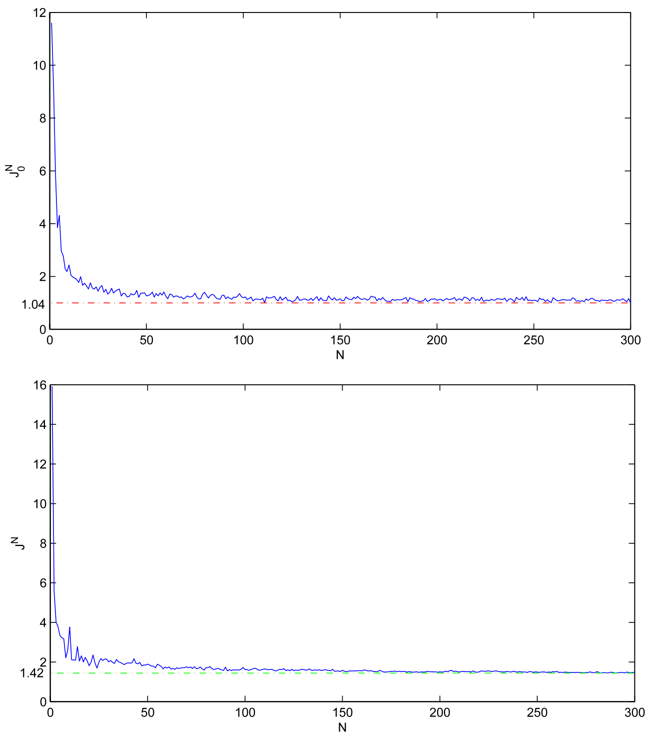

Fig. 2. Trajectories of J_0^N and $J^N = \max_{1 \le i \le N} J_i^N$ with respect to N.

Noticing $H = 0.5$ and

$$M = \begin{pmatrix} 0.5 & 0.2 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \times \begin{pmatrix} 0.93 & 0 \\ 0 & 0.81 \end{pmatrix},$$

it can be verified that Assumption (A4) holds. From (22) and (23) we can get the following distributed control laws:

$$u_0^*(k) = (1 - F_0(\theta_k))g^*(k) + 5 - 3x_0(k), \tag{57}$$

$$u_i^*(k) = (0.5 - F(\theta_k))g_i^*(k) - 2x_i(k) - 0.05x_0(k) + \alpha_i, \quad 1 \le i \le N, \tag{58}$$

where

$$g^*(k) = 0.2x_0(k+1) + 0.5g^*(k) + 2, \qquad g^*(0) = 1,$$
$$g_i^*(k) = 0.2x_0(k+1) + 0.5g_i^*(k) + \alpha_i, \qquad g_i^*(0) = x_{i0}.$$

We first consider the consistency of MF estimation. When the number of agents is 300, the trajectories of $x^{(N)}$, g^* and g_i^*, $i = 1, \ldots, 300$, are shown as in Fig. 1. By Theorem 4.1, the error between the major agent's MF estimation function g^* and the state average of all the minor agents $x^{(N)}$ should converge to 0 in the average sense as N grows to ∞. From Fig. 1, it can be seen that g^* almost coincides with $x^{(300)}$, which illustrates the consistency of MF estimation. On the other hand, due to the influence of the initial value x_{i0} and the parameters α_i, the trajectories of the minor agent's MF estimation functions g_i^*, $i = 1, \ldots, 300$, fluctuate around g^*.

Then we check the index values of all the agents under the distributed control laws (57) and (58). Let $J^N = \max_{1 \le i \le N} J_i^N$. Then, by $\delta = 0.09$ and Theorem 4.3, J_0^N and J^N are at most $\sum_{j=1}^{2} \pi_j D_0^2(j) = 1.04$ and $J^N = \sum_{j=1}^{2} \pi_j D_j^2 + (1-|H|)^{-2} \sum_{j=1}^{m} \pi_j |F(j) - H|^2 \delta = 1.42$, respectively. When the number of agents grows from 1 to 300, the trajectories of J_0^N and J^N are shown as in Fig. 2, from which one can be see that the index values tend to the upper bounds 1.04 and 1.42.

We now consider the influence of α_i on the indices of the minor agents. For the model (53)–(56), except that the variance δ of the parameters α_i, $i = 1, \ldots, N$, is 0 (i.e., $\alpha_i \equiv 2$), all the parameters are unchanged. When the number of agents grows from 1 to 300, the trajectory of $J^N = \max_{1 \le i \le N} J_i^N$ is shown as in Fig. 3. It can be seen that when the variance of the parameter α_i is 0 and N grows to ∞, the maximum of the indices of all the minor agents J^N tends to $0.68 (= \sum_{j=1}^{2} \pi_j D_j^2)$, which is the optimal index value of the centralized control.

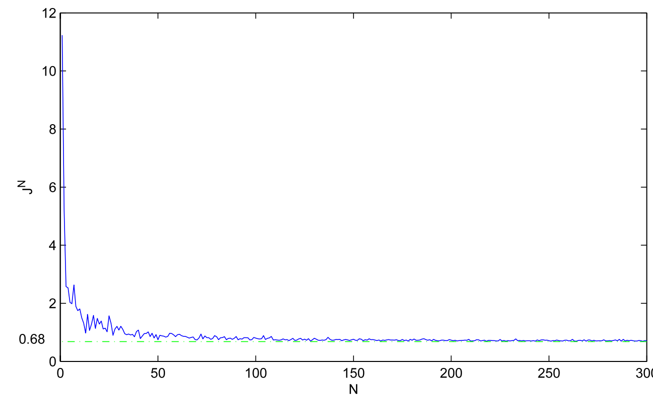

Fig. 3. Trajectories $J^N = \max_{1 \le i \le N} J_i^N$ with respect to N when $\delta = 0$.

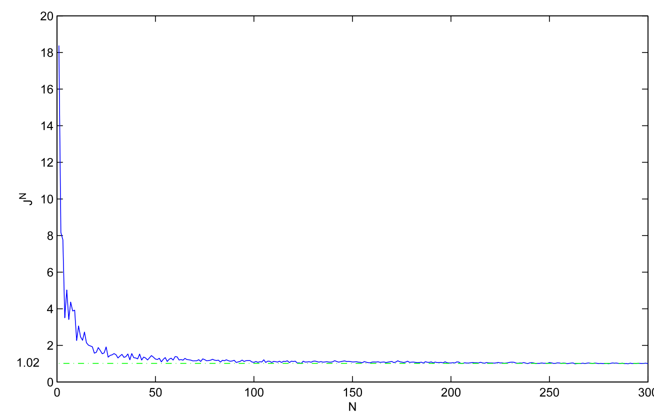

Fig. 4. Trajectory of $J^N = \max_{1 \le i \le N} J_i^N$ with respect to N when $x_{i0} \sim N(1, 0.09)$.

Finally, we consider the influence of the initial values x_{i0} on the indices of the minor agents for (53)–(56) in the case of $\bar{H}_0 = 0, H = 1$ and $\alpha_i \equiv 0$. We take two classes of initial values $\{x_{i0}, i = 1, \ldots, N\}$ as example. One is that $\{x_{i0}, i = 1, \ldots, N\}$ is i.i.d. and with $x_{i0} \sim N(1, 0.09)$, and the other is constant $x_{i0} \equiv 1$. For the first class of initial values, we have $\delta_0 = 0.09$, and hence, by Theorem 5.2 the maximum of the indices of the minor agents J^N should tend to

$$\sum_{j=1}^{2} \pi_j D_j^2 + \sum_{j=1}^{2} \pi_j |F(j) - 1|^2 \delta_0$$
$$= \sum_{j=1}^{2} \pi_j D_j^2 + 0.09 \times \sum_{j=1}^{2} \pi_j |F(j) - 1|^2 = 1.02$$

as $N \to \infty$, and for the second class of initial values, we have $\delta_0 = 0$, and the maximum of the indices should tend to $\sum_{j=1}^{2} \pi_j D_j^2 + \sum_{j=1}^{2} \pi_j |F(j) - 1|^2 \delta_0 = \sum_{j=1}^{2} \pi_j D_j^2 = 0.68$. From Figs. 4 and 5, we can see that the upper bounds of the index values are achieved as N grows to ∞.

7. Concluding remarks

In this paper, we investigate the distributed control problem of large population MASs involving a major agent. There are Markov

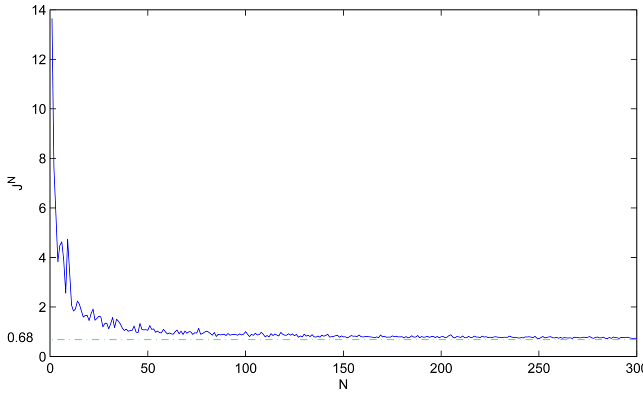

Fig. 5. Trajectory of $J^N = \max_{1\le i\le N} J_i^N$ with respect to N when $x_{i0} \equiv 1$.

jump parameters in the dynamics and random parameters in index functions. Except the state of the major player, each agent only knows the information of its state and parameters. For the quadratic tracking indices, by using the MF theory, a set of feasible distributed control is presented. The uniform stability and sub-optimality of the closed-loop systems are proved.

Although the model (1)–(4) looks like a Stackelberg game, they are essentially different. In Stackelberg games, the leaders are in a dominant place and have the superiority to make decision first, and then the followers give the strategies sequentially. However, the agents in the model (1)–(4) are equal and can make their decisions simultaneously, no matter how high or low each agent's influence is.

There are many problems worthy of investigating in this area, including the distributed controls with energy constraints or actuator saturation, the case with networked communications among the agents, the strategy design and time-inconsistency issues in Stackelberg games for MAS with dominant agents etc. (Kydland, 1977; Kydland & Prescott, 1977).

Appendix

Proof of Theorem 3.1. From Assumptions (A1), (A2) and (3) we have

$$\begin{aligned}
J_0^N(u) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\Big\|f_0(\theta_k, x_0(k)) + u_0(k) + F_0(\theta_k)x^{(N)}(k) \\
&\quad - H_0x^{(N)}(k) - \alpha_0 + D_0(\theta_k)w_0(k+1)\Big\|^2 \\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\Big\{E\|f_0(\theta_k, x_0(k)) + u_0(k) - \alpha_0 \\
&\quad + (F_0(\theta_k) - H_0)x^{(N)}(k)\|^2 + E\|D_0(\theta_k)w_0(k+1)\|^2 \\
&\quad + 2E\Big\{w_0^T(k+1)[f_0(\theta_k, x_0(k)) + u_0(k) \\
&\quad + (F_0(\theta_k) - H_0)x^{(N)}(k) - \alpha_0]\Big\}\Big\} \\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\{E\|f_0(\theta_k, x_0(k)) + u_0(k) - \alpha_0 \\
&\quad + (F_0(\theta_k) - H_0)x^{(N)}(k)\|^2 + E\|D_0(\theta_k)w_0(k+1)\|^2\}.
\end{aligned}$$

Since $\{\theta_k\}$ is ergodic and $\{\theta_k\}$ is independent of $\{w_0(k)\}$, we can get

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\|D_0(\theta_k)w_0(k+1)\|^2 \\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\{\mathrm{tr}[D_0(\theta_k)w_0(k+1)w_0^T(k+1)D_0^T(\theta_k)]\} \\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} \mathrm{tr}\{E[D_0^T(\theta_k)D_0(\theta_k)] \\
&\quad \times E[w_0(k+1)w_0^T(k+1)]\} \\
&= \sum_{j=1}^{m}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\mathrm{tr}(D_0^T(j)D_0(j))p_j(k) \\
&= \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0^T(j)D_0(j)),
\end{aligned}$$

where $p_j(k) = P(\theta_k = j)$. Hence,

$$\begin{aligned}
J_0^N(u) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\|f_0(\theta_k, x_0(k)) + u_0(k) \\
&\quad + [F_0(\theta_k) - H_0]x^{(N)}(k) - \alpha_0\|^2 \\
&\quad + \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0^T(j)D_0(j)) \\
&\ge \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0^T(j)D_0(j)).
\end{aligned} \tag{A.1}$$

By (A1), (A2) and (4) we have

$$\begin{aligned}
J_i^N(u) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\Big\|f_i(\theta_k, x_i(k)) + u_i(k) \\
&\quad + (F(\theta_k) - H)x^{(N)}(k) + G(\theta_k)x_0(k) \\
&\quad - \bar{H}_0x_0(k) - \alpha_i + D(\theta_k)w_i(k+1)\Big\|^2 \\
&= \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\Big\{E\Big\|f_i(\theta_k, x_i(k)) + u_i(k) \\
&\quad + (F(\theta_k) - H)x^{(N)}(k) + G(\theta_k)x_0(k) - \bar{H}_0x_0(k) - \alpha_i\Big\|^2 \\
&\quad + E\|D(\theta_k)w_i(k+1)\|^2\Big\} \\
&\ge \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_jD_j^T).
\end{aligned} \tag{A.2}$$

Take

$$\begin{aligned}
\bar{u}_0(k) &= (H_0 - F_0(\theta_k))x^{(N)}(k) + \alpha_0 - f_0(\theta_k, x_0(k)), \\
\bar{u}_i(k) &= (\bar{H}_0 - G(\theta_k))x_0(k) + (H - F(\theta_k))x^{(N)}(k) \\
&\quad + \alpha_i - f_i(\theta_k, x_i(k)), \quad 1 \le i \le N.
\end{aligned}$$

Then, by the definition of $\mathcal{U}_{g,i}$, we have $\bar{u}_i \in \mathcal{U}_{g,i}$, $0 \le i \le N$, and by (A.1) and (A.2), the corresponding index values are

$$J_0^N(\bar{u}) = \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_0(j)D_0^T(j)).$$

$$J_i^N(\bar{u}) = \sum_{j=1}^{m}\pi_j\mathrm{tr}(D_jD_j^T). \quad \square$$

Proof of Lemma 1. Since $(P^T \otimes I_{n^2})\,\mathrm{diag}\{A_i \otimes A_i\}$ is stable, by (Costa et al., 2005) we know that

$$N_i = A_i^T\sum_{j=1}^{m}p_{ij}N_jA_i + I_n, \quad i, j = 1, \ldots, m$$

has a unique set of definite solutions $N_i,\ i = 1, \dots, m$. Thus, by the Markov property of $\{\theta_k, X_k\}$, we have

$$\begin{aligned}
&E[X_{k+1}^T N(\theta_{k+1})X_{k+1}] \\
&\quad = \sum_{j=1}^m E[X_{k+1}^T N(\theta_{k+1})X_{k+1}I_{[\theta_{k+1}=j]}] \\
&\quad = \sum_{j=1}^m E[X_k^T A^T(\theta_k)N_j A(\theta_k)X_k I_{[\theta_{k+1}=j]}] \\
&\qquad + \sum_{j=1}^m E[W_{k+1}^T D_j^T N_j D_j W_{k+1} I_{[\theta_{k+1}=j]}] \\
&\quad = \sum_{j=1}^m E\left\{E[X_k^T A^T(\theta_k)N_j A(\theta_k)X_k I_{[\theta_{k+1}=j]}|\mathcal{G}_k]\right\} \\
&\qquad + \sum_{j=1}^m p_j(k+1)E[W_{k+1}^T D_j^T N_j D_j W_{k+1}] \\
&\quad = E\left[X_k^T A^T(\theta_k)\sum_{j=1}^m p_{\theta_k j}N_j A(\theta_k)X_k\right] \\
&\qquad + \sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j) \\
&\quad = E[X_k^T N(\theta_k)X_k] - E[X_k^T X_k] \\
&\qquad + \sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j),
\end{aligned} \tag{A.3}$$

where $\mathcal{G}_k = \sigma(X_l, \theta_l, l \le k)$, $N(j) = N_j$. Summing the both sides of the above equation from $k = 0$ to T gives

$$\begin{aligned}
&E[X_{T+1}^T N(\theta_{T+1})X_{T+1}] \\
&\quad = E[X_0^T N(\theta_0)X_0] - \sum_{k=0}^T E\|X_k\|^2 \\
&\qquad + \sum_{k=0}^T \sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j).
\end{aligned}$$

Since $\|N_j\| < \infty$ and $\{\theta_k\}$ is ergodic, we obtain

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^T E\|X_k\|^2 \\
&\quad = \limsup_{T\to\infty} \frac{1}{T}\left\{E[X_0^T N_{\theta_0}X_0] - E[X_{T+1}^T N_{\theta_{T+1}}X_{T+1}]\right. \\
&\qquad \left. + \sum_{k=0}^T\sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j)\right\} \\
&\quad \le \limsup_{T\to\infty} \frac{1}{T}\left\{E[X_0^T N(\theta_0)X_0]\right. \\
&\qquad \left. + \sum_{k=0}^T\sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j)\right\} \\
&\quad = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^T\sum_{j=1}^m p_j(k+1)\mathrm{tr}(rD_j^T N_j D_j) \\
&\quad = \sum_{j=1}^m \pi_j \mathrm{tr}(rD_j^T N_j D_j) \le C_0 r\sum_{j=1}^m \pi_j \mathrm{tr}(D_j^T D_j). \quad \square
\end{aligned} \tag{A.4}$$

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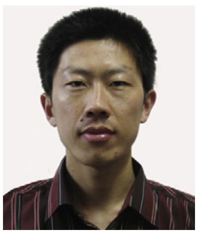

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